Lecture 3: Affine Schemes

Note Title 6/20/2019

$A =$ commutative algebra w/ a unit.

$\rightsquigarrow$ $\mathrm{Spec}A :=$ the set of prime ideals of A

$\mathfrak{a} \subseteq A$ ideal $\rightsquigarrow$ $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}A \mid \mathfrak{p} \supseteq \mathfrak{a}\}$

Topology of SpecA

Proposition 1: ① $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$

② $V(\sum \mathfrak{a}_i) = \cap V(\mathfrak{a}_i)$

③ $V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \iff \sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$

Define the closed subsets of SpecA are of the form $V(\mathfrak{a})$

The proposition implies that this defines a topology on SpecA.

In particular, $V(0) = \mathrm{Spec}A$, $V(A) = \phi$

pf: ① $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b} \iff \mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$

② $\mathfrak{p} \supseteq \sum \mathfrak{a}_i \iff \mathfrak{p} \supseteq \mathfrak{a}_i, \forall i$

③ $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}$

$f \in A$. then $D(f) := \mathrm{Spec}A \setminus V((f))$ defines an open set of SpecA

Proposition 2: $\{D(f)\}_{f \in A}$ forms a base of above topology.

pf: Claim: $\forall \mathfrak{a} \trianglelefteq A$, $\exists f \in A$ s.t $\underline{D(f) \subseteq \mathrm{Spec}A \setminus V(\mathfrak{a})}$

$\iff D(f) \cap V(\mathfrak{a}) = \phi$

$\nexists\ \underline{\mathfrak{p} \notin V(\mathfrak{a})},\ \underline{\mathfrak{p} \in D(f)}$

$\Updownarrow \qquad \Updownarrow$

$\mathfrak{a} \not\subseteq \mathfrak{p} \Rightarrow \exists f \notin \mathfrak{p}.\ f \in \mathfrak{a}$

Sheaf of Rings $\mathcal{O}_{\mathrm{Spec}A}$ on $\mathrm{Spec}A$

$U \underset{\text{open}}{\subseteq} \mathrm{Spec}A$ $\qquad \mathcal{O}(U) \ni s : U \longrightarrow \coprod_{\mathfrak{p}\in U} \underbrace{A_{\mathfrak{p}}}_{\text{localization of } A \text{ at } \mathfrak{p}}$

$\mathfrak{p} \in \underset{\text{open}}{V} \subseteq U$, $\mathfrak{q} \in V$

$s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}}$, $f \notin \mathfrak{q}$, $\forall \mathfrak{q} \in V$

Check: $\mathcal{O}_{\mathrm{Spec}A}$ defines a sheaf on $\mathrm{Spec}A$.

Definition: $(\mathrm{Spec}A, \underbrace{\mathcal{O}_{\mathrm{Spec}A}}_{\text{structure sheaf}})$ is called an <u>affine scheme</u>

Proposition 3: ① $\mathcal{O}_{\mathfrak{p}} \cong A_{\mathfrak{p}}$, $\forall \mathfrak{p} \in \mathrm{Spec}A$

② $\mathcal{O}(D(f)) \cong A_f$, $\forall f \in A$

③ $\Gamma(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A}) \cong A$

pf: ① define $A_{\mathfrak{p}} \xrightarrow{\varphi} \mathcal{O}_{\mathfrak{p}}$

$\frac{a}{f} \mapsto s(\mathfrak{p}) = \frac{a}{f}$ on $V \ni \mathfrak{p}$

$f \notin \mathfrak{q}, \forall \mathfrak{q} \in V$

surjectivity: trivial

injectivity:

$\frac{a}{f} = \frac{a'}{f'}$

$\exists h \notin \mathfrak{p}$ s.t. $h(af' - a'f) = 0$

$af'h - a'fh \Rightarrow s(\mathfrak{p}) = s'(\mathfrak{p})$

on their domain $\cap D(h)$

② $\Rightarrow$ ③ by taking $f = 1$

② define $A_f \xrightarrow{\psi} \mathcal{O}(D(f))$

$\frac{a}{f^n} \longmapsto s(\mathfrak{p}) = \frac{a}{f^n} \in A_{\mathfrak{p} \in D(f)}$

injectivity: Otherwise $\exists\ \psi\left(\frac{a}{f^n}\right) = \psi\left(\frac{b}{f^m}\right)$

i.e $\exists h \notin \mathfrak{p}$ s.t $h(af^m - bf^n) = 0 \in A$, $\forall \mathfrak{p} \in D(f)$

$\mathfrak{a} =$ annihilator of $(af^m - bf^n)$

$h \in \mathfrak{a}$ $\Longrightarrow$ $\mathfrak{a} \not\subseteq \mathfrak{p}$, $\forall \mathfrak{p} \in D(f)$ or $V(\mathfrak{a}) \cap D(f) = \phi$

$\Longrightarrow$ $f \in \sqrt{\mathfrak{a}}$ or $f^l \in \mathfrak{a}$ for some l

Prop 1 ③

$\Longrightarrow$ $f^l(af^m - bf^n) = 0 \in A$

i.e $\frac{a}{f^n} = \frac{b}{f^m} \in A_f$

Surjectivity: $s \in D(f)$ i.e $\exists\ V_i \subseteq D(f)$ open cover

s.t $s = \frac{a_i}{g_i} \in A_{\mathfrak{p}}$, $\forall \mathfrak{p} \in V_i$

- $\{D(h)\}$ forms bases $\Longrightarrow$ $\exists h_i \in A$ s.t $D(h_i) \subseteq V_i \subseteq D(g_i)$

$\Updownarrow$

$h_i^{n_i} = c_i g_i$

By replacing $\frac{a_i}{g_i} = \frac{c_i a_i}{h_i^{n_i}}$, may assume $V_i = D(h_i)$, $s = \frac{a_i}{h_i}$ on V_i

- $D(f) = \overset{\text{finit}}{\bigcup} D(h_i)$

$D(f) = \bigcup D(h_i) \Longleftrightarrow \bigcap V(h_i) \subseteq V(f) \Longleftrightarrow f^n = \underbrace{\sum b_i h_i}_{\text{finite}}$

$\bigcap V(h_i) = V(\Sigma(h_i))$

- $D(h_i) \cap D(h_j) = D(h_i h_j)$

injectivity $\Longrightarrow$ $\dfrac{a_i}{h_i} = \dfrac{a_j}{h_j}$ on $D(h_i h_j)$

$n_{ij} = n$ since only finitely pair

i.e $(h_i h_j)^{n_{ij}} (a_i h_j - a_j h_i) = 0 \in A$

$$\Updownarrow$$

$$(a_i h_i^n)\, h_j^{n+1} = (a_j h_j^n)\, h_i^{n+1}$$

$$a_i \quad h_j \qquad a_j \quad h_i$$

May assume $s = \dfrac{a_i}{h_i}$ on $D(h_i)$ & $a_i h_j = a_j h_i$

- Write $b = \sum b_i a_i$

$$b h_j = \sum b_i a_i h_j = \sum b_i h_i a_j = f^n a_j$$

$$\frac{b}{f^n} = \frac{a_j}{h_j} \quad \text{on } D(h_j) \cap D(f)$$

Definition: $(X, \mathcal{O}_X)$ $\underline{\text{local ringed space}}$ if X topological space, $\mathcal{O}_X$ sheaf of local rings

$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$ morphism of local ringed spaces

if $X \xrightarrow{f} Y$ continuous, $\mathcal{O}_Y \xrightarrow{f^\#} f_* \mathcal{O}_X$ morphism of rings

$$f_p^\# : \mathcal{O}_{Y, f(p)} \longrightarrow \mathcal{O}_{X,p}$$

$$(f_p^\#)^{-1}(m_{\mathcal{O}_{X,p}}) = m_{\mathcal{O}_{Y,f(p)}}$$

Proposition 4: $\left\{ \begin{array}{c} \text{Commutative algebras} \\ \text{w/ unit} \end{array} \right\} \underset{\text{equivalence of categories}}{\cong} \left\{ \text{affine schemes} \right\}$

pf: • $A \xrightarrow{\varphi} B$ homomorphism

$\rightsquigarrow (f, f^\#) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

$$f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$
$$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$$

$$V(\varphi(\mathfrak{a})) = f^{-1}(V(\mathfrak{a})) \implies f \text{ is continuous}$$

$\mathfrak{p} \supseteq \varphi(\mathfrak{a}) \iff \underset{\overset{\shortparallel}{f(\mathfrak{p})}}{\varphi^{-1}(\mathfrak{p})} \supseteq \mathfrak{a}$

• $\mathcal{O}_{\operatorname{Spec} A} \xrightarrow{f^\#} f_*(\mathcal{O}_{\operatorname{Spec} B})$ $A_{\varphi^{-1}(\mathfrak{p})} \xrightarrow{\varphi_\mathfrak{p}} B_\mathfrak{p}, \ \forall \mathfrak{p} \triangleleft B$ prime

gluing

$$\mathcal{O}_{\operatorname{Spec} A}(U) \longrightarrow f_*(\mathcal{O}_{\operatorname{Spec} B})(U) = \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(U))$$

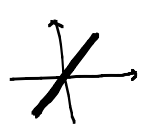
• Conversely, given $(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \xrightarrow{(f, f^\#)} (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ locally ringed

$\longrightarrow$ take global section

$$\begin{array}{ccc} A \cong \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) & \xrightarrow{f^\#} & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \cong B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} \cong \mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} & \longrightarrow & \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}} \cong B_\mathfrak{p} \end{array}$$

$$f^\# \text{ local homomorphism} \implies \varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$$

ex. $\mathbb{A}^1_k \xhookrightarrow{i}$



$$k[t] \cong k[x,y]/(x-y) \overset{\varphi}{\twoheadleftarrow} k[x,y]$$
$$(0) \overset{i}{\longmapsto} \varphi^{-1}(0) = (x-y)$$

generic point sent to generic point of the image

Definition: A <u>scheme</u> is a locally ringed space w/ open cover $\{U_i\}$ s.t U_i is isomorphic to an affine scheme.

ex. $\operatorname{Spec} k$, k = field

ex. $\operatorname{Spec} k[x,y]$, closed points $\longleftrightarrow$ k^2

"fat points" $\longleftrightarrow$ w/ closure algebraic curves $\subseteq k^2$
generic points

or k^2

$R := k[x]$ is a PID, UFD

Fact: prime ideals of $R[y]$ are of the form

1. (0) generic point of $\mathbb{A}^2_k$
2. $(f(y))$, $f(y)$ irreducible in $R[y]$ generic point of $\{F(x,y)=0\}$
3. $(p, f(y))$, $p \in R$ irreducible, $f(y)$ irreducible in $R/p[y]$
closed point

ex. R = discrete valuation ring, K = quotient field of R
$\cup$
m : maximal ideal

$\operatorname{Spec} R = \{(0), m\}$

$(0) \subseteq m \implies \overline{(0)} = \operatorname{Spec} R$, m is a closed point

$R_{(0)} \cong K$ $\qquad R_m \cong R$

$$R \hookrightarrow K \rightsquigarrow \begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} R \\ & \searrow & \cup \\ & & (0) \end{array}$$

Gluing of schemes

X_1, X_2 schemes

$$\begin{array}{cc} X_1 & X_2 \\ {\scriptstyle i_1}\uparrow & \uparrow{\scriptstyle i_2} \\ U_1 & \cong U_2 \end{array}$$

$\mathcal{O}_{X_1}(U_1) \overset{\varphi}{\cong} \mathcal{O}_{X_2}(U_2)$

$X = X_1 \cup X_2$ topologically

$$\mathcal{O}_X(V) = \left\{ (s_1, s_2) \in \mathcal{O}_{X_1}(i_1^{-1}(V)) \times \mathcal{O}_{X_2}(i_2^{-1}(V)) \;\middle|\; \varphi(s_1|_{i_1^{-1}(V)}) = s_2|_{i_2^{-1}(V)} \right\}$$

ex. $\operatorname{Spec} k[x] = \mathbb{A}^1_k$, glue two pieces of $\mathbb{A}^1$'s along $\mathbb{A}^1 \setminus \{0\}$

$\rightsquigarrow$ non-seperated scheme, non-affine scheme.

- $\exists$ two points not seperated via open subsets
- For affine scheme, $q \subseteq \overline{p} \Leftrightarrow p \supseteq q$

A generalization of previous Proposition 4.

Proposition 5: A = ring, $(X, \mathcal{O}_X)$ = scheme $\Rightarrow$ $\operatorname{Hom}_{Sch}(X, \operatorname{Spec} A) \cong \operatorname{Hom}_{ring}(A, \Gamma(X, \mathcal{O}_X))$

pf: $\operatorname{Hom}_{Sch}(X, \operatorname{Spec} A) \ni f \rightsquigarrow \mathcal{O}_{\operatorname{Spec} A} \xrightarrow{f^\#} f_*\mathcal{O}_X$

$\xrightarrow[\text{take global section}]{} A \xrightarrow{f^\#} \Gamma(X, \mathcal{O}_X) \in \operatorname{Hom}_{ring}(A, \Gamma(X, \mathcal{O}_X))$

Proposition holds as X is affine.

In general, cover X by affine charts U_i

$U_i \cap U_j = U_{ijk}$

$$0 \to \Gamma(X, \mathcal{O}_X) \to \prod_i \Gamma(U_i, \mathcal{O}_{U_i}) \to \prod_{i,j,k} \Gamma(U_{ijk}, \mathcal{O}_{U_{ijk}}) \quad \text{exact}$$

$$\{s_i\} \longmapsto \{s_i - s_j|_{U_{ijk}}\}$$

$\rightsquigarrow$ taking $\operatorname{Hom}(A, -)$

$$0 \to \operatorname{Hom}(A, \Gamma(X, \mathcal{O}_X)) \to \prod_i \operatorname{Hom}(A, \Gamma(U_i, \mathcal{O}_{U_i})) \to \prod_{i,j,k} \operatorname{Hom}(A, \Gamma(U_{ijk}, \mathcal{O}_{U_{ijk}}))$$

$\cong$ as sets $\cong$

from gluing of schemes $0 \to \operatorname{Hom}(X, \operatorname{Spec} A) \to \prod_i \operatorname{Hom}(U_i, \operatorname{Spec} A) \to \prod_{i,j,k} \operatorname{Hom}(U_{ijk}, \operatorname{Spec} A)$

Projective schemes

$S = \bigoplus_{d \geq 0} S_d$ graded algebra

$S_d \cdot S_e \subseteq S_{d+e}$

$\mathfrak{a} \triangleleft S$ is homogeneous if $\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a} \cap S_d$

Equivalently, $\mathfrak{a}$ generated by homog. elements

$S_+ := \bigoplus_{d > 0} S_d$

Define $\mathrm{Proj}(S) :=$ set of prime ideals $\mathfrak{p}$ of S not containing S_+.

$\Updownarrow$

ab homog. then $ab \in \mathfrak{p}$ iff $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$

$\mathfrak{a}$: homog. ideal. $V(\mathfrak{a}) = \{ \mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{p} \supseteq \mathfrak{a} \}$

defines the closed subsets of $\mathrm{Proj}(S)$.

$\mathfrak{p}$: homog. ideal $\rightsquigarrow S_{(\mathfrak{p})} = \left\{ \frac{a}{b} \,\middle|\, \underset{\text{homog.}}{a, b \in S}, \deg\left(\frac{a}{b}\right) = 0 \ b \notin \mathfrak{p} \right\}$

$$\mathcal{O}(\mathcal{U}) \ni s : \underset{\text{open}}{\mathcal{U}} \longrightarrow \coprod_{\mathfrak{p} \in \mathcal{U}} S_{(\mathfrak{p})}$$

$$\mathfrak{p} \in V \subseteq \mathcal{U} \quad (\exists V) \qquad s(\mathfrak{q}) = \frac{a}{f \notin} \in S_{(\mathfrak{q})} \text{ on } V, \ \forall \mathfrak{q} \in V$$

a, f homog. of same degree

$$\rightsquigarrow (\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$$

Proposition 5: ① $(\mathcal{O}_{\mathrm{Proj}(S)})_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$ from definition of $\mathcal{O}_{\mathrm{Proj}(S)}$

② $D_+(f) := \{ \mathfrak{p} \in \mathrm{Proj}\, S \mid f \notin \mathfrak{p} \}$ open subset, $\underset{\text{homog.}}{f \in S_+}$

$$(D_+(f), \mathcal{O}_{\mathrm{Proj}(S)}|_{D_+(f)}) \cong \mathrm{Spec}\, S_{(f)}$$

② $\Rightarrow$ ③ $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a scheme.

pf: $\underset{\text{prime}}{\mathfrak{p}} \triangleleft S_{(f)} = \left\{ \frac{s}{f^k} \;\middle|\; s \in S,\ \deg(s) = k\deg(f) \right\}$

$\Downarrow$

$\tilde{\mathfrak{p}} = \left\{ s \in S \;\middle|\; \frac{s}{f^k} \in S_{(f)},\ \text{for some } k \in \mathbb{N} \cup \{0\} \right\}$

generated by homog. elements

if $\underset{\text{homog}}{a, b \in S}$, $ab \in \tilde{\mathfrak{p}}$, then
$$\begin{array}{ccc} S & \xrightarrow{\phi} & S_{(f)} \\ ab & \longmapsto & \phi(ab) \in \mathfrak{p} \end{array}$$

$\therefore\ \phi(a) \in \mathfrak{p}$ or $\phi(b) \in \mathfrak{p}$

$a \in \tilde{\mathfrak{p}}$ or $b \in \tilde{\mathfrak{p}}$

i.e. $\tilde{\mathfrak{p}} \in \mathrm{Proj}(S)$, $\tilde{\mathfrak{p}} \not\ni f$

Thus, $D_+(f) \underset{\text{top.}}{\cong} \mathrm{Spec}(S_{(f)})$

$D_+(f)$ base of topology on Proj S

Claim: $S_{(f)} \xrightarrow{\simeq} \mathcal{O}_{\mathrm{Proj}(S)}(D_+(f))$ $\Rightarrow$ $\mathcal{O}_{\mathrm{Spec}\, S_{(f)}} \cong \mathcal{O}_{\mathrm{Proj}(S)}\big|_{D_+(f)}$

$\frac{s}{f^k} \longmapsto \left(\mathfrak{p} \mapsto \frac{s}{f^k} \in S_{(\mathfrak{p})} \right)$

Proof is similar to Proposition 3 ②

ex. k = algebraically closed field

$\mathbb{P}^n_k := \mathrm{Proj}\, k[x_0, \cdots, x_n]$

For affine scheme $X = \mathrm{Spec} A$, $\Gamma(X, \mathcal{O}_X) = A$ global functions recover the geometry

Q: Similar results for $X = \mathrm{Proj}(S)$?

Proposition 6: $k =$ algebraically closed

then $\mathrm{Var}(k) \hookrightarrow \mathrm{Sch}(k)$ fully faithful.

pf: • $X =$ topological space $\rightsquigarrow$ $t(X)$ set of irreducible closed subset of X

closed subsets are of the form

$t(X) \supseteq t(Y)$ for $\underset{\text{closed}}{Y \subseteq X}$

$t(Y_1) \cup t(Y_2) = t(Y_1 \cup Y_2)$, $t(\cap Y_i) = \cap t(Y_i)$

topology on $t(Y)$: closed subsets $\| $ $t(Y)$, $\underset{\text{closed}}{Y \subseteq X}$

$f: X_1 \longrightarrow X_2$ continuous map

$\rightsquigarrow \tilde{f}: t(X_1) \longrightarrow t(X_2)$

$\overline{P} \mapsto \overline{f(P)}$ irreducible

$\alpha: X \longrightarrow t(X)$

$P \longmapsto \overline{P}$

$\underset{\text{open}}{V \subseteq X} \xleftrightarrow{1:1} \underset{\text{open}}{t(\alpha(V)^c)^c \subseteq t(X)}$

Now $V =$ variety w/ ring of regular functions $\mathcal{O}_V$

Claim: $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme.

$\cong$ $(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$

it suffices to show the case V affine say w/ coordinate ring A.

irreducible algebraic subset of V $\xleftrightarrow{1:1}$ prime ideals of A $\rightsquigarrow t(V) \underset{\text{homeo}}{\cong} \mathrm{Spec}A$

(closed) points of V $\xleftrightarrow{1:1}$ maximal $=$

$$(V, \mathcal{O}_V) \longrightarrow (\overset{X}{\overset{\|}{\mathrm{Spec} A}}, \mathcal{O}_{\mathrm{Spec} A})$$ morphism of ringed spaces as follows

$$\mathcal{O}_X(U) \xrightarrow{\cong} \alpha_* \mathcal{O}_V(U) = \mathcal{O}_V(\alpha^{-1}(U))$$

$$s \longmapsto \left(p \longmapsto s(p) \in \mathcal{O}_{X, \alpha(p)} \cong A_{m_p} \longrightarrow A_{m_p}/m_p \cong k \right)$$

with $s \in \mathcal{O}_X(U)$, $p \in \alpha^{-1}(U)$; regular function

- $(t(V), \alpha_* \mathcal{O}_V)$ is defined $/ k$ — it suffices to have the natural map $k \longrightarrow \Gamma(V, \mathcal{O}_V)$

- $\mathrm{Hom}_{Var(k)}(V, W) = \mathrm{Hom}_{Sch(k)}(t(V), t(W))$

  via gluing of schemes & morphisms, it suffices to consider the case when both V & W are affine